

DIAGONAL APPROXIMATION IN COMPLETIONS OF \mathbb{Q}

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ABSTRACT. We prove analogues of some classical results from Diophantine approximation and metric number theory (namely Dirichlet's theorem and the Duffin-Schaeffer theorem) in the setting of diagonal Diophantine approximation, i.e. approximating elements of $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r}$ by elements of the diagonal embedding of \mathbb{Q} into this space.

1. INTRODUCTION

In this paper, we prove analogues of some theorems of Diophantine approximation and metric number theory in the new setting of diagonal approximation. As motivation, we begin with a brief overview of the results we intend to prove analogues of, and of some of the analogues which have already been proven.

It is a theorem, dating to 1842 and due to Dirichlet, that for any real number x and natural number N , there exists some $a \in \mathbb{Z}$ and $n \in \mathbb{N}$ with $n \leq N$ satisfying

$$|nx - a| \leq \frac{1}{N}.$$

A corollary of this says that for any irrational x , there exist infinitely many coprime $a \in \mathbb{Z}, n \in \mathbb{N}$ satisfying

$$|nx - a| \leq \frac{1}{n}.$$

There have been versions of this theorem proved in the simultaneous approximation setting (where we approximate elements of \mathbb{R}^m by $a \in \mathbb{Z}^m$ and $n \in \mathbb{N}$; see §1.5 of [1]) and in the p -adic setting (see [7]). In §2, we extend these results to the diagonal setting, which can be seen as a combination of the results in these two settings.

Our aim is to find a natural method of approximating elements of $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r}$ (where the p_i are *different* primes). To justify our approach, we note that the theorems of classical approximation, which were originally theorems about \mathbb{R} , can also be viewed as theorems about \mathbb{R}/\mathbb{Z} (or its canonical fundamental domain $[0, 1)$). We quotient our space by a lattice which is natural to the problem (in the classical case, we quotient \mathbb{R} by \mathbb{Z}), and end up with a compact measure space. This makes it much easier to state results from metric number theory, about the measure of the sets satisfying certain properties.

Given this, we want to find such a lattice in $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r}$. Consider the space

$$\mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right] = \{ np_1^{\nu_1} \cdots p_r^{\nu_r} \mid n, \nu_1, \dots, \nu_r \in \mathbb{Z} \}.$$

(Elements β and γ of this space will always be given the decompositions

$$(1) \quad \beta = mp_1^{\mu_1} \cdots p_r^{\mu_r} \quad \text{and} \quad \gamma = np_1^{\nu_1} \cdots p_r^{\nu_r},$$

where m and n are each coprime to all of the p_i .

This space is a subset of \mathbb{Q} , which is in turn a subset of \mathbb{R} and of each of the \mathbb{Q}_{p_i} . So we can look at the image $\iota\left(\mathbb{Z}\left[\frac{1}{p_1}, \dots, \frac{1}{p_r}\right]\right)$ of this space under the diagonal embedding

$$\begin{aligned} \iota : \quad \mathbb{Q} &\longrightarrow \mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r} \\ \gamma &\longmapsto (\gamma, \gamma, \dots, \gamma). \end{aligned}$$

This image is a cocompact lattice in $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r}$, and the quotient space

$$(\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r}) / \iota\left(\mathbb{Z}\left[\frac{1}{p_1}, \dots, \frac{1}{p_r}\right]\right)$$

can be identified with the fundamental domain

$$[0, 1) \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}.$$

We then approximate elements of this fundamental domain by elements of

$$\iota\left(\mathbb{Z}\left[\frac{1}{p_1}, \dots, \frac{1}{p_r}\right]\right).$$

We define a distance function on $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r}$ by

$$d(\mathbf{x}, \mathbf{y}) = \max\{|x_\infty - y_\infty|_\infty, |x_{p_1} - y_{p_1}|_{p_1}, \dots, |x_{p_r} - y_{p_r}|_{p_r}\},$$

and a function on $\mathbb{Z}\left[\frac{1}{p_1}, \dots, \frac{1}{p_r}\right]$ by

$$\ell(\gamma) = \max\{|\gamma|_\infty, |\gamma|_{p_1}, \dots, |\gamma|_{p_r}\},$$

which we call the level of γ . Using these notions, we can prove the following result.

Theorem 1.1. *Let $\mathbf{x} \in \mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r}$, and let $N \in \mathbb{N}$. Then there exists some $\beta, \gamma \in \mathbb{Z}\left[\frac{1}{p_1}, \dots, \frac{1}{p_r}\right]$ with $\ell(\gamma) \leq N$ and $\gamma > 0$ satisfying*

$$d(\gamma\mathbf{x}, \beta) \leq \frac{P}{N},$$

where $P = \max\{p_1, \dots, p_r\}$.

(Note that when we write $d(\gamma\mathbf{x}, \beta)$, what we strictly mean is $d(\gamma\mathbf{x}, \iota(\beta))$.)

We will also prove an analogue of the corollary to Dirichlet's theorem given above.

We then move on to the main result of this paper, which is an analogue of the classical Duffin–Schaeffer theorem (Thm I from [2]). The classical result says that for a function $\psi : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ satisfying

$$\sum_{n=1}^{\infty} \psi(n) = \infty$$

and

$$(2) \quad \limsup_{N \rightarrow \infty} \frac{\sum_{n=1}^N \frac{\psi(n)\varphi(n)}{n}}{\sum_{n=1}^N \psi(n)} > 0$$

(where $\varphi(n)$ is the Euler totient function), the set of those $x \in [0, 1]$ which satisfy

$$|nx - a| < \psi(n)$$

for infinitely many coprime a, n is of Lebesgue measure 1.

The long-standing Duffin–Schaeffer conjecture states that this theorem should be true even without condition (2), and Pollington and Vaughan (see [8]) proved the natural simultaneous approximation analogue of this conjecture in \mathbb{R}^k for $k \geq 2$.

To state our theorem, we start out with a function $\psi : \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right] \rightarrow \mathbb{R}_{\geq 0}$, and restrict it so that

$$(3) \quad \psi(\gamma) \leq \frac{1}{2L(\gamma)},$$

where

$$L(\gamma) = \max_{i \in \{\infty, p_1, \dots, p_r\}} \frac{|\gamma|_{\infty} |\gamma|_{p_1} \cdots |\gamma|_{p_r}}{|\gamma|_i} = \max_{i \in \{\infty, p_1, \dots, p_r\}} \frac{n}{|\gamma|_i}.$$

(This restriction corresponds to the (implicit) condition in Duffin and Schaeffer’s original paper ([2]) that $\psi(n) \leq \frac{1}{2}$, and ensures that for a given γ and \mathbf{x} , the distance $d(\gamma\mathbf{x}, \beta)$ is only less than $\psi(\gamma)$ for at most one β .)

For this function ψ , we define a set $\mathcal{A}(\psi)$ by

$$(4) \quad \mathcal{A}(\psi) = \left\{ \mathbf{x} \in [0, 1) \times \prod_{p_i} \mathbb{Z}_{p_i} \mid \begin{array}{l} d(\gamma\mathbf{x}, \beta) < \psi(\gamma) \text{ for infinitely} \\ \text{many coprime } \beta, \gamma \in \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right] \end{array} \right\}.$$

(What it means for two elements of $\mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right]$ to be coprime will be explained in §2.)

Finally, we fix the measure on $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r}$ to be the product measure of Lebesgue measure on \mathbb{R} and normalised Haar measure on the \mathbb{Q}_{p_i} .

Then our theorem is as follows.

Theorem 1.2. *If we have*

$$(5) \quad \sum_{\gamma \in \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right]} \frac{\varphi(n)\psi(\gamma)^{r+1}}{n} = \infty$$

and

$$\limsup_{N \rightarrow \infty} \frac{\sum_{\ell(\gamma) \leq N} \frac{\varphi(n)\psi(\gamma)^{r+1}}{n}}{\sum_{\ell(\gamma) \leq N} \psi(\gamma)^{r+1}} > 0,$$

then $\mathcal{A}(\psi)$ has measure 1.

There is also a result of Haynes [5, Theorem 4] which gives a simultaneous approximation analogue in $\mathbb{R}^\ell \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_k}$. While this result looks similar to our Theorem 1.2, they are distinct results, as we are approximating by elements of a diagonal embedding.

In §3, we will first show a partial converse to Theorem 1.2, that convergence of the sum in (5) implies that $\mathcal{A}(\psi)$ is of measure 0. This will follow almost directly from the convergence part of the Borel–Cantelli lemma, as in the classical case.

In the final three sections, we first develop some of the machinery required to prove Theorem 1.2, and then conclude by proving the theorem. In §4, we prove the following zero-one law, which is an analogue to Theorem 1 in [3].

Theorem 1.3. *For each $\psi : \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right] \rightarrow \mathbb{R}_{\geq 0}$ which satisfies (3), the set $\mathcal{E}(\psi)$ of those $\mathbf{x} \in [0, 1) \times \prod_{i=1}^r \mathbb{Z}_{p_i}$ such that*

$$d(\gamma \mathbf{x}, \beta) \leq \psi(\gamma)$$

for infinitely many coprime $\beta, \gamma \in \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right]$ has measure 0 or 1.

In §5, we prove a technical lemma (Lemma 5.1), which provides estimates for the measure of the overlap between certain sets.

Finally, in §6 we use Theorem 1.3 and Lemma 5.1 to prove Theorem 1.2.

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2. DIRICHLET’S THEOREM IN THE DIAGONAL SETTING

Proof of Theorem 1.1. Consider the points of the form

$$\zeta \mathbf{x} - \iota(\beta_\zeta),$$

where ζ ranges over all elements of $\mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right]$ with $\ell(\zeta) \leq N$ and $\zeta \geq 0$, and the $\beta_\zeta \in \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right]$ are chosen so that the points lie in $[0, 1) \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$. This can be done uniquely since $[0, 1) \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$ is a fundamental domain for our quotient space

$$(\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r}) / \iota \left(\mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right] \right).$$

If any two of these points are equal, then taking their difference yields β and γ such that $d(\gamma \mathbf{x}, \beta) = 0$. So we may assume they are all distinct. To apply the pigeonhole argument we want to use, we need to know how many points of this form there are. Since they are all distinct, this is equivalent to calculating the size of the set

$$Z_N = \left\{ \zeta \in \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right] \mid \ell(\zeta) \leq N, \zeta \geq 0 \right\}.$$

For each $i = 1, \dots, r$, we can find $n_i \in \mathbb{Z}_{\geq 0}$ such that

$$p_i^{n_i} \leq N < p_i^{n_i+1}.$$

So since $|\cdot|_p$ takes discrete values from $\{p^m \mid m \in \mathbb{Z}\}$, we want

$$0 \leq |\zeta|_\infty \leq N \quad \text{and} \quad 0 \leq |\zeta|_{p_i} \leq p_i^{n_i}$$

for each i .

The p_i -adic conditions tell us we are dealing with a subset of $\frac{1}{p_1^{n_1} \cdots p_r^{n_r}} \mathbb{Z}$, and combining this with the first condition, we get

$$\#Z_N = N p_1^{n_1} \cdots p_r^{n_r} + 1.$$

Now, let $P = \max\{p_1, \dots, p_r\}$, and let η denote the corresponding n_i (that is, if $P = p_j$, then $\eta = n_j$). Then we have $N > P^\eta$, and hence

$$\#Z_N > P^\eta p_1^{n_1} \cdots p_r^{n_r}.$$

Now consider the boxes of the form

$$\left[\frac{r}{P^\eta}, \frac{r+1}{P^\eta} \right) \times (s_1 + p_1^{n_1} \mathbb{Z}_{p_1}) \times \cdots \times (s_r + p_r^{n_r} \mathbb{Z}_{p_r}),$$

where $0 \leq r \leq P^\eta - 1$ and $0 \leq s_i \leq p_i^{n_i} - 1$.

Since we have $P^\eta p_1^{n_1} \cdots p_r^{n_r}$ boxes which cover our fundamental domain, and we have

$$\{\zeta \mid \ell(\zeta) \leq N\} > P^\eta p_1^{n_1} \cdots p_r^{n_r},$$

there will be two elements (corresponding to ζ and ξ , say) in one box. So the distance between them is bounded above by the diameter of the box, which is given by

$$\max_{i=1, \dots, r} \frac{1}{p_i^{n_i}}.$$

Since for any p_i we have $N < P p_i^{n_i}$, this is bounded above by $\frac{P}{N}$. Then, assuming WLOG that $|\zeta|_\infty > |\xi|_\infty$, we define $\gamma = \zeta - \xi$ and $\beta = \beta_\zeta - \beta_\xi$. Then we have $\ell(\gamma) \leq N$ and

$$d(\gamma \mathbf{x}, \beta) \leq \frac{P}{N}$$

as required. \square

We now prove a corollary of this theorem, which is an analogue of the corollary to Dirichlet's theorem given in the introduction. To do this, we first need to note what it means for two elements $\beta, \gamma \in \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right]$ to be coprime.

Let β, γ be elements of this space, and decompose them as in (1). Then we say that β and γ are coprime if m and n are coprime in the usual sense, and define $\gcd(\beta, \gamma) := \gcd(m, n)$. This definition of coprimality comes from the fact that $\mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right]$ is a UFD; by adjoining the inverses of each of the p_i , we have made them into units, and hence we are justified in ignoring them as factors of β and γ .

Now we state our corollary.

Corollary 2.1. *Let $\mathbf{x} \in (\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r}) - \iota(\mathbb{Q})$. Then there exist infinitely many coprime $\beta, \gamma \in \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right]$ such that*

$$d(\gamma \mathbf{x}, \iota(\beta)) \leq \frac{1}{\ell(\gamma)}.$$

Proof of Corollary 2.1. For each $n \in \mathbb{N}$, let $B_n, C_n \in \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_r}]$ be such that $\ell(C_n) \leq n$ and

$$d(C_n \mathbf{x}, B_n) \leq \frac{P}{n}.$$

Let $\beta_n = \frac{B_n}{\gcd(B_n, C_n)}$ and $\gamma_n = \frac{C_n}{\gcd(B_n, C_n)}$. Then we have

$$d(\gamma_n \mathbf{x}, \beta_n) = \frac{1}{\gcd(B_n, C_n)} d(C_n \mathbf{x}, B_n) \leq d(C_n \mathbf{x}, B_n) \leq \frac{P}{n} \leq \frac{p}{\ell(C_n)} \leq \frac{P}{\ell(\gamma_n)}.$$

So now we just need to show that of the (β_n, γ_n) , infinitely many are distinct. But suppose that there are only finitely many distinct pairs

$$(\beta_{n_1}, \gamma_{n_1}), \dots, (\beta_{n_m}, \gamma_{n_m}),$$

and consider

$$C := \min_{i=1, \dots, m} d(\gamma_{n_i} \mathbf{x}, \beta_{n_i}).$$

If $C = 0$, then for some (β_n, γ_n) we have

$$d(\gamma_n \mathbf{x}, \beta_n) = 0.$$

But this can only happen when $\mathbf{x} \in \iota(\mathbb{Q})$, and we assumed otherwise.

However, if $C > 0$, then take N to be some integer with $N > \frac{P}{C}$, and consider (β_N, γ_N) . We have

$$d(\gamma_N \mathbf{x}, \beta_N) \leq \frac{P}{N} < C,$$

giving a contradiction. \square

3. A PARTIAL CONVERSE TO THEOREM 1.2

Before we prove Theorem 1.2, we will prove the following result.

Lemma 3.1. *Let $\psi : \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_r}] \rightarrow \mathbb{R}_{\geq 0}$ be such that $\psi(\gamma) \leq \frac{1}{2L(\gamma)}$ and such that the sum*

$$\sum_{\gamma \in \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_r}]} \frac{\varphi(n) \psi(\gamma)^{r+1}}{n}$$

converges. Then the set $\mathcal{A}(\psi)$ (as defined in (4)) has measure 0.

In the classical case, this follows almost directly from the convergence part of the Borel–Cantelli lemma. In the diagonal case, we need to do a little work first, in order to rewrite our set $\mathcal{A}(\psi)$ as a limsup of sets $A_\gamma(\psi)$ and then estimate the measure of these sets.

Proof. We note that

$$\begin{aligned}
d(\gamma \mathbf{x}, \beta) < \psi(\gamma) &\Leftrightarrow \max_{p \in \{\infty, p_1, \dots, p_r\}} |\gamma x_p - \beta|_p < \psi(\gamma) \\
&\Leftrightarrow |\gamma x_p - \beta|_p < \psi(\gamma) \text{ for each } p \in \{\infty, p_1, \dots, p_r\} \\
&\Leftrightarrow \left| x_p - \frac{\beta}{\gamma} \right|_p < \frac{\psi(\gamma)}{|\gamma|_p} \text{ for each } p \in \{\infty, p_1, \dots, p_r\} \\
&\Leftrightarrow \left| x_\infty - \frac{\beta}{\gamma} \right|_\infty < \frac{\psi(\gamma)}{np_1^{\nu_1} \dots p_r^{\nu_r}} \\
&\quad \text{and } \left| x_{p_i} - \frac{\beta}{\gamma} \right|_{p_i} < p_i^{\nu_i} \psi(\gamma) \text{ for } i = 1, \dots, r.
\end{aligned}$$

So define

$$(6) \quad \mathcal{A}_\gamma(\psi) := \bigcup_{\substack{\beta \in \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_r}] \\ (\beta, \gamma)=1}} \left(B_\infty \left(\frac{\beta}{\gamma}, \frac{\psi(\gamma)}{np_1^{\nu_1} \dots p_r^{\nu_r}} \right) \times \prod_{i=1}^r B_{p_i} \left(\frac{\beta}{\gamma}, p_i^{\nu_i} \psi(\gamma) \right) \right).$$

Then if we consider an ordering γ_i of $\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_r}]$ such that if $i \leq j$, then $\ell(\gamma_i) \leq \ell(\gamma_j)$ (that is, we order by level), then we have

$$\mathcal{A}(\psi) = \limsup_{\ell(\gamma) \rightarrow \infty} \mathcal{A}_\gamma(\psi) := \limsup_{i \rightarrow \infty} \mathcal{A}_{\gamma_i}(\psi).$$

We now claim that our expression for $\mathcal{A}_\gamma(\psi)$ can be simplified to

$$\mathcal{A}_\gamma(\psi) = \bigcup_{\substack{a=1 \\ (a, n)=1}}^n \left(B_\infty \left(\frac{a}{n}, \frac{\psi(\gamma)}{np_1^{\nu_1} \dots p_r^{\nu_r}} \right) \times \prod_{i=1}^r B_{p_i} \left(\frac{a}{n}, p_i^{\nu_i} \psi(\gamma) \right) \right).$$

First, note that (6) can be rewritten as

$$\begin{aligned}
&\bigcup_{\substack{a \in \mathbb{Z} \\ (a, p_1 \dots p_r n)=1}} \bigcup_{\mu_1 \in \mathbb{Z}} \dots \bigcup_{\mu_r \in \mathbb{Z}} \left(B_\infty \left(\frac{ap_1^{\mu_1} \dots p_r^{\mu_r}}{np_1^{\nu_1} \dots p_r^{\nu_r}}, \frac{\psi(\gamma)}{np_1^{\nu_1} \dots p_r^{\nu_r}} \right) \right. \\
&\quad \left. \times \prod_{i=1}^r B_{p_i} \left(\frac{ap_1^{\mu_1} \dots p_r^{\mu_r}}{np_1^{\nu_1} \dots p_r^{\nu_r}}, p_i^{\nu_i} \psi(\gamma) \right) \right).
\end{aligned}$$

Now we want to show that the boxes where $\mu_i < \nu_i$ for some $i = 1, \dots, r$ or where $ap_1^{\mu_1 - \nu_1} \dots p_r^{\mu_r - \nu_r} \notin [0, n]$ do not intersect our fundamental domain $[0, 1) \times \mathbb{Z}_{p_1} \times \dots \times \mathbb{Z}_{p_r}$.

Suppose that $\mu_i < \nu_i$. Then we have

$$\frac{ap_1^{\mu_1} \dots p_r^{\mu_r}}{np_1^{\nu_1} \dots p_r^{\nu_r}} = \frac{a}{p_i^\sigma n} \prod_{\substack{j=1 \\ j \neq i}}^r p_j^{\mu_j - \nu_j}$$

for some $\sigma \in \mathbb{N}$. This is not in \mathbb{Z}_{p_i} . We also have

$$p_i^{\nu_i} \psi(\gamma) < |\gamma|_{p_i}^{-1} \frac{1}{L(\gamma)} \leq |\gamma|_{p_i}^{-1} \frac{|\gamma|_{p_i}}{n} = \frac{1}{n} \leq 1.$$

So our p_i -adic ball is centred on something not in \mathbb{Z}_{p_i} , and has radius < 1 . This means it cannot intersect \mathbb{Z}_{p_i} , and therefore any box with $\mu_i < \nu_i$ cannot intersect $[0, 1) \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$.

So suppose that for each i we have $\mu_i \geq \nu_i$, but then $ap_1^{\mu_1 - \nu_1} \cdots p_r^{\mu_r - \nu_r} \notin [0, n]$. Then since $ap_1^{\mu_1 - \nu_1} \cdots p_r^{\mu_r - \nu_r}$ is an integer, the distance from $ap_1^{\mu_1 - \nu_1} \cdots p_r^{\mu_r - \nu_r}$ to the set $[0, n]$ must be at least 1, and hence the distance from $\frac{ap_1^{\mu_1} \cdots p_r^{\mu_r}}{np_1^{\nu_1} \cdots p_r^{\nu_r}}$ to the set $[0, 1]$ must be at least $\frac{1}{n}$. But we know that

$$\frac{\psi(\gamma)}{|\gamma|_\infty} < \frac{1}{2n},$$

and hence $B_\infty\left(\frac{ap_1^{\mu_1} \cdots p_r^{\mu_r}}{np_1^{\nu_1} \cdots p_r^{\nu_r}}, \frac{\psi(\gamma)}{np_1^{\nu_1} \cdots p_r^{\nu_r}}\right)$ cannot intersect $[0, 1]$. So any box with

$$ap_1^{\mu_1 - \nu_1} \cdots p_r^{\mu_r - \nu_r} \notin [0, n]$$

cannot intersect $[0, 1) \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$, and so we have our claim. Now we want to calculate the measure of $\mathcal{A}_\gamma(\psi)$. We have

$$\lambda(\mathcal{A}_\gamma(\psi)) = \varphi(n) \cdot \lambda\left(B_\infty\left(\frac{a}{n}, \frac{\psi(\gamma)}{|\gamma|_\infty}\right)\right) \cdot \prod_{i=1}^r \lambda\left(B_{p_i}\left(\frac{a}{n}, \frac{\psi(\gamma)}{|\gamma|_{p_i}}\right)\right).$$

Then since

$$\lambda\left(B_\infty\left(\frac{a}{n}, \frac{\psi(\gamma)}{|\gamma|_\infty}\right)\right) = \frac{2\psi(\gamma)}{|\gamma|_\infty}$$

and

$$\frac{\psi(\gamma)}{p_i |\gamma|_{p_i}} < \lambda\left(B_{p_i}\left(\frac{a}{n}, \frac{\psi(\gamma)}{|\gamma|_{p_i}}\right)\right) \leq \frac{\psi(\gamma)}{|\gamma|_{p_i}},$$

we have that

$$\frac{2\varphi(n)\psi(\gamma)^{r+1}}{np_1 \cdots p_r} < \lambda(\mathcal{A}_\gamma) \leq \frac{2\varphi(n)\psi(\gamma)^{r+1}}{n}.$$

So if

$$\sum_{\gamma \in \mathbb{Z}\left[\frac{1}{p_1}, \dots, \frac{1}{p_r}\right]} \frac{\varphi(n)\psi(\gamma)^{r+1}}{n} < \infty,$$

then we have that

$$\sum_{i=1}^{\infty} \lambda(\mathcal{A}_{\gamma_i}) = \sum_{\gamma \in \mathbb{Z}\left[\frac{1}{p_1}, \dots, \frac{1}{p_r}\right]} \lambda(\mathcal{A}_\gamma) < \infty$$

as well. So by the convergence part of the Borel–Cantelli lemma, $\mathcal{A}(\psi)$ has measure 0. \square

Now we turn our attention to the proof of our main theorem.

4. THE ZERO-ONE LAW

In this section, we prove Theorem 1.3. For this, we will need a preliminary lemma, which is a direct analogue of Lemma 2 from [3].

Lemma 4.1. *Let $\{I_k\}$ be a sequence of boxes in $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r}$ such that $\lambda(I_k) \rightarrow 0$ (where a box is a product of a ball from each of the constituent spaces), and let $\{U_k\}$ be a sequence of measurable sets (also in $\mathbb{R} \times \mathbb{Q}_{p_1} \times \cdots \times \mathbb{Q}_{p_r}$) such that, for some positive $\varepsilon < 1$, we have*

$$U_k \subset I_k \quad \text{and} \quad \lambda(U_k) \geq \varepsilon \lambda(I_k).$$

Then

$$\lambda \left(\limsup_{k \rightarrow \infty} I_k \right) = \lambda \left(\limsup_{k \rightarrow \infty} U_k \right).$$

Proof. We define

$$\begin{aligned} \mathcal{I} &= \bigcap_{L=1}^{\infty} \bigcup_{k \geq L} I_k \left(= \limsup_{k \rightarrow \infty} I_k \right), \\ \mathcal{U}_k &= \bigcup_{n \geq k} U_n, \\ \mathcal{D}_k &= \mathcal{I} - \mathcal{U}_k. \end{aligned}$$

Then the lemma can be restated as

$$\lambda \left(\bigcup_{k=1}^{\infty} \mathcal{D}_k \right) = 0.$$

We prove that each \mathcal{D}_k has measure 0.

We say that a point $\mathbf{x} \in [0, 1) \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$ is a density point if each of its coordinates $x_\infty, x_{p_1}, \dots, x_{p_r}$ are density points in their respective spaces. By the Lebesgue density theorem, and its analogue in \mathbb{Q}_p (see p14 in [6]), almost all points \mathbf{x} are density points.

So now suppose for a contradiction that x_0 is a density point of \mathcal{D}_k in \mathcal{D}_k .

Firstly, since we have that $x_0 \in I_k$ for infinitely many k and that $\lambda(I_k) \rightarrow 0$, if we restrict to those k such that $x_0 \in I_k$, we have

$$\lambda(\mathcal{D}_k \cap I_k) \sim \lambda(I_k)$$

as $k \rightarrow \infty$ (since x_0 is a density point).

However, we also have that $\mathcal{D}_k \cap U_k = \emptyset$, and hence U_k and $\mathcal{D}_k \cap I_k$ are disjoint subsets of I_k . From this, we get

$$\lambda(I_k) \geq \lambda(U_k) + \lambda(\mathcal{D}_k \cap I_k) \geq \varepsilon \lambda(I_k) + \lambda(\mathcal{D}_k \cap I_k),$$

and hence

$$\lambda(\mathcal{D}_k \cap I_k) \leq (1 - \varepsilon) \lambda(I_k),$$

contradicting our first part. \square

Now we prove our zero-one law.

Proof of Theorem 1.3. For each prime q greater than the product $p_1 \cdots p_r$, and for each $\nu \in \mathbb{N}$, we define

$$\mathfrak{A}(q^\nu) = \{\mathbf{x} \mid d(\gamma\mathbf{x}, \beta) < q^{\nu-1}\psi(\gamma) \text{ for infinitely many coprime } \beta, \gamma \text{ with } q \nmid \gamma\}$$

and

$$\mathfrak{B}(q^\nu) = \{\mathbf{x} \mid d(\gamma\mathbf{x}, \beta) < q^{\nu-1}\psi(\gamma) \text{ for infinitely many coprime } \beta, \gamma \text{ with } q \parallel \gamma\},$$

where $q \parallel \gamma$ means that $q \mid \gamma$ but $q^2 \nmid \gamma$.

Note that both $\mathfrak{A}(q)$ and $\mathfrak{B}(q)$ are subsets of $\mathcal{E}(\psi)$ for each prime q . Note also that we have $\mathfrak{A}(q^\nu) \subseteq \mathfrak{A}(q^{\nu+1})$ and $\mathfrak{B}(q^\nu) \subseteq \mathfrak{B}(q^{\nu+1})$ for all $\nu \in \mathbb{N}$.

Now, we know that

$$\lambda \left(B_\infty \left(\frac{b}{n}, \frac{\psi(\gamma)}{bp_1^{\nu_1} \cdots p_r^{\nu_r}} \right) \times \prod_{i=1}^r B_{p_i} \left(\frac{b}{n}, p_i^{\nu_i} \psi(\gamma) \right) \right) \sim \frac{\psi(\gamma)^{r+1}}{n} \rightarrow 0$$

as $\ell(\gamma) \rightarrow \infty$ (this follows from $\psi(\gamma) \leq \frac{1}{2L(\gamma)}$). So by Lemma 4.1 we have that $\lambda(\mathfrak{A}(q^\nu)) = \lambda(\mathfrak{A}(q))$ for all $\nu \in \mathbb{N}$, and hence (since they form a chain) the union $\mathfrak{A}^*(q)$ of the $\mathfrak{A}(q^\nu)$ also has measure $\lambda(\mathfrak{A}(q))$.

The same argument shows that the union $\mathfrak{B}^*(q)$ of the $\mathfrak{B}(q^\nu)$ has measure $\lambda(\mathfrak{B}(q))$.

Now we construct maps T_A and T_B such that

$$T_A(\mathfrak{A}^*(q)) \subseteq \mathfrak{A}^*(q) \quad \text{and} \quad T_B(\mathfrak{B}^*(q)) \subseteq \mathfrak{B}^*(q).$$

Suppose that \mathbf{x} has

$$d(\gamma\mathbf{x}, \beta) < q^{\nu-1}\psi(\gamma)$$

for β, γ coprime with $q \nmid \gamma$. (that is, suppose $\mathbf{x} \in \mathfrak{A}(q^\nu)$, presuming this happens infinitely often).

This means that

$$|\gamma x_\infty - \beta|_\infty < q^{\nu-1}\psi(\gamma) \quad \text{and} \quad |\gamma x_{p_i} - \beta|_{p_i} < q^{\nu-1}\psi(\gamma) \text{ for } i = 1, \dots, r.$$

Then we have

$$\left| \gamma \left(\frac{q}{p_1 \cdots p_r} x_\infty \right) - \left(\frac{\beta q}{p_1 \cdots p_r} \right) \right|_\infty = \left| \frac{q}{p_1 \cdots p_r} \right|_\infty q^{\nu-1}\psi(\gamma) = \frac{q^\nu}{p_1 \cdots p_r} \psi(\gamma) < q^\nu \psi(\gamma)$$

and

$$\left| \gamma \left(\frac{q}{p_1 \cdots p_r} x_{p_i} \right) - \left(\frac{\beta q}{p_1 \cdots p_r} \right) \right|_{p_i} = p_i q^{\nu-1} \psi(\gamma) < q^\nu \psi(\gamma).$$

We have

$$\left(\frac{\beta q}{p_1 \cdots p_r}, \gamma \right) = (\beta q, \gamma) = (\beta, \gamma) = 1,$$

where the first equality comes from the fact that our notion of coprime excludes powers of any of the p_i , and the second comes from our assumption that $q \nmid \gamma$. So

$$\mathbf{x} \mapsto \frac{q}{p_1 \cdots p_r} \mathbf{x} \pmod{\iota \left(\mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right] \right)}$$

sends $\mathfrak{A}(q^\nu)$ to $\mathfrak{A}(q^{\nu+1})$, and thus takes $\mathfrak{A}^*(q)$ into itself. Denote this map by T_A .

Via exactly the same argument, we can show that the map T_B given by

$$\mathbf{x} \mapsto \frac{q}{p_1 \cdots p_r} \mathbf{x} + \frac{p_1 \cdots p_r}{q}$$

sends $\mathfrak{B}^*(q)$ to itself.

We now show that for any integer $q > p_1 \cdots p_r$ and any integer s , the map T defined by

$$\mathbf{x} \mapsto \frac{q}{p_1 \cdots p_r} \mathbf{x} + \frac{sp_1 \cdots p_r}{q} \pmod{\iota \left(\mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right] \right)}$$

is metrically transitive on $[0, 1) \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$. That is to say, for any set A such that $T(A) \subseteq A$, we have that A has either measure 0 or 1.

Now assume that A has positive measure, and that $T(A) \subseteq A$. This implies that $T^n(A) \subseteq A$ for all $n \in \mathbb{N}$. Let $\phi_A(\mathbf{x})$ be the indicator function for A . Then we have

$$\phi(\mathbf{x}) \leq \phi(T^n(\mathbf{x})).$$

Let \mathbf{x}_0 be a density point of A , and consider the sets

$$\mathfrak{I}_n = \mathfrak{I}_{\infty, n} \times \prod_{i=1}^r \mathfrak{I}_{p_i, n},$$

where $\mathfrak{I}_{\infty, n}$ is the open interval of length $\left(\frac{p_1 \cdots p_r}{q}\right)^n$ centred on the real coordinate of \mathbf{x}_0 , and $\mathfrak{I}_{p_i, n}$ is a p_i -adic ball of measure p_i^{-n} centred on the p_i -adic coordinate of \mathbf{x}_0 .

Then we have

$$\begin{aligned} \lambda(A \cap \mathfrak{I}_n) &= \int_{\mathfrak{I}_n} \phi(\mathbf{x}) \, d\mathbf{x} \\ &\leq \int_{\mathfrak{I}_n} \phi(T^n(\mathbf{x})) \, d\mathbf{x} \\ &= \int_{\mathfrak{I}_{p_1, n}} \cdots \int_{\mathfrak{I}_{p_r, n}} \int_{\mathfrak{I}_{\infty, n}} \phi \left(\left(\frac{q}{p_1 \cdots p_r} \right)^n \mathbf{x} + \frac{sp_1 \cdots p_r}{q} \right) \, dx_\infty dx_{p_r} \cdots dx_{p_1}. \end{aligned}$$

Let

$$\mathbf{y} = \left(\frac{q}{p_1 \cdots p_r} \right)^n \mathbf{x} + \frac{sp_1 \cdots p_r}{q}.$$

Then we have

$$y_\infty = \left(\frac{q}{p_1 \cdots p_r} \right)^n x_\infty + \frac{sp_1 \cdots p_r}{q}, \quad y_{p_i} = \left(\frac{q}{p_i} \right)^n x_{p_i} + \frac{sp_1 \cdots p_r}{q},$$

and hence

$$dx_\infty = \left(\frac{p_1 \cdots p_r}{q} \right)^n dy_\infty, \quad dx_{p_i} = \frac{dy_{p_i}}{p_i^n}.$$

So

$$\begin{aligned} & \int_{\mathfrak{I}_{p_1, n}} \cdots \int_{\mathfrak{I}_{p_r, n}} \int_{\mathfrak{I}_{\infty, n}} \phi \left(\left(\frac{q}{p_1 \cdots p_r} \right)^n \mathbf{x} + \frac{sp_1 \cdots p_r}{q} \right) dx_\infty dx_{p_r} \cdots dx_{p_1} \\ &= \int_{\frac{q^n}{(p_1 \cdots p_r)^n} \mathfrak{I}_{p_1, n} + \frac{sp_1 \cdots p_r}{q}} \int_{\left(\frac{q}{p_1 \cdots p_r} \right)^n \mathfrak{I}_{\infty, n} + \frac{sp_1 \cdots p_r}{q}} \phi(\mathbf{y}) \frac{1}{q^n} dy_\infty dy_{p_r} \cdots dy_{p_1} \\ &= \frac{1}{q^n} \int_{\mathbb{Z}_{p_1}} \cdots \int_{\mathbb{Z}_{p_r}} \int_{(0,1)} \phi(\mathbf{y}) d\mathbf{y}, \end{aligned}$$

and hence

$$\lambda(A \cap \mathfrak{I}_n) \leq \lambda(\mathfrak{I}_n) \lambda(A).$$

So in the limit, $\lambda(A) \geq 1$, and hence $\lambda(A) = 1$. So if we have $\lambda(A) > 0$, we have $\lambda(A) = 1$, and hence T is metrically transitive.

Since $\mathfrak{A}^*(q)$ and $\mathfrak{B}^*(q)$ both go into themselves under a map of this form (T_A and T_B respectively) we conclude that they are both either of measure 0 or 1, and hence that both $\mathfrak{A}(q)$ and $\mathfrak{B}(q)$ are.

Now, if either $\mathfrak{A}(q)$ or $\mathfrak{B}(q)$ is of measure 1 for any prime $q > p_1 \cdots p_r$, we know that $\mathcal{E}(\psi)$ is measure 1 (since $\mathfrak{A}(q)$ and $\mathfrak{B}(q)$ are both subsets of $\mathcal{E}(\psi)$). So now assume that $\mathfrak{A}(q)$ and $\mathfrak{B}(q)$ both have measure 0 for all primes $q > p_1 \cdots p_r$. For each of those primes, define a set $\mathfrak{C}(q)$ by

$$\mathfrak{C}(q) = \{\mathbf{x} \mid d(\gamma\mathbf{x}, \beta) < \psi(\gamma) \text{ for infinitely many coprime } \beta, \gamma \text{ with } q^2 \mid \gamma\}.$$

Since we assumed that $\mathfrak{A}(q)$ and $\mathfrak{B}(q)$ both have measure 0, we have $\lambda(\mathcal{E}(\psi)) = \lambda(\mathfrak{C}(q))$.

Note that if we have

$$d(\gamma\mathbf{x}, \beta) < \psi(\gamma)$$

with $(\beta, \gamma) = 1$ and $q^2 \mid \gamma$, then we have

$$d\left(\gamma \left(\mathbf{x} \pm \frac{1}{q}\right), \beta \pm \frac{\gamma}{q}\right) = d(\gamma\mathbf{x}, \beta) < \psi(\gamma).$$

So $\mathfrak{C}(q)$ has “period” $\iota(\frac{1}{q})$. It then follows, since $\mathfrak{C}(q)$ and $\mathcal{E}(\psi)$ differ by a set of measure 0, that for each set

$$\mathcal{I}_q := I_q \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r},$$

where I_q is an real interval of length $\frac{1}{q}$, we have

$$\lambda(\mathcal{E}(\psi) \cap \mathcal{I}_q) = \lambda(\mathcal{I}_q) \lambda(\mathcal{E}(\psi)) = \frac{\lambda(\mathcal{E}(\psi))}{q}.$$

Suppose that $\lambda(\mathcal{E}(\psi)) > 0$, and let \mathbf{x}_0 be a density point of $\mathcal{E}(\psi)$. Consider the sequence $\{\mathcal{I}_q\}$, where the real interval part is centred at the real coordinate of \mathbf{x}_0 . By our version of the Lebesgue density theorem, we have that

$$\lambda(\mathcal{E}(\psi) \cap \mathcal{I}_q) \sim \lambda(\mathcal{I}_q)$$

as $q \rightarrow \infty$. So we have $\mathcal{E}(\psi) = 1$, completing the proof. \square

5. OVERLAP ESTIMATES

In this section, we prove the following result, which is analogous to Lemma II in [2].

Lemma 5.1. *Let β and γ be elements of $\mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_r}]$, and write*

$$\beta = mp_1^{\mu_1} \cdots p_r^{\mu_r} \quad \text{and} \quad \gamma = np_1^{\nu_1} \cdots p_r^{\nu_r}.$$

Define

$$\mathcal{A}_\beta = \bigcup_{\substack{a=1 \\ (a,m)=1}}^m \left(B_\infty \left(\frac{a}{m}, \frac{\psi(\beta)}{mp_1^{\mu_1} \cdots p_r^{\mu_r}} \right) \times \prod_{i=1}^r B_{p_i} \left(\frac{a}{m}, p_i^{\mu_i} \psi(\beta) \right) \right),$$

and define \mathcal{A}_γ in the same way. Then we have

$$\lambda(\mathcal{A}_\beta \cap \mathcal{A}_\gamma) \leq 16\psi(\beta)^{r+1}\psi(\gamma)^{r+1}.$$

Proof. To get an upper bound for the measure of $\mathcal{A}_\beta \cap \mathcal{A}_\gamma$, we note that each set is made up of a union of disjoint boxes. We then sum over all pairs of boxes which intersect, with the summand being an upper bound for the measure of their intersection.

A pair of boxes will intersect if and only if their projections into each given completion (which will be a pair of intervals) also intersect. So for the overlap between

$$B_\infty \left(\frac{a}{m}, \frac{\psi(\beta)}{mp_1^{\mu_1} \cdots p_r^{\mu_r}} \right) \times \prod_{i=1}^r B_{p_i} \left(\frac{a}{m}, p_i^{\mu_i} \psi(\beta) \right)$$

and

$$B_\infty \left(\frac{b}{n}, \frac{\psi(\gamma)}{np_1^{\nu_1} \cdots p_r^{\nu_r}} \right) \times \prod_{i=1}^r B_{p_i} \left(\frac{b}{n}, p_i^{\nu_i} \psi(\gamma) \right)$$

to be of positive measure, we want the real overlap

$$B_\infty \left(\frac{a}{m}, \frac{\psi(\beta)}{mp_1^{\mu_1} \cdots p_r^{\mu_r}} \right) \cap B_\infty \left(\frac{b}{n}, \frac{\psi(\gamma)}{np_1^{\nu_1} \cdots p_r^{\nu_r}} \right)$$

to be of positive measure, and for each p_i -adic overlap

$$B_{p_i} \left(\frac{a}{m}, p_i^{\mu_i} \psi(\beta) \right) \cap B_{p_i} \left(\frac{b}{n}, p_i^{\nu_i} \psi(\gamma) \right)$$

to also be of positive measure.

The real intervals will overlap when

$$\left| \frac{a}{m} - \frac{b}{n} \right|_\infty \leq 2 \max \left\{ \frac{\psi(\beta)}{mp_1^{\mu_1} \cdots p_r^{\mu_r}}, \frac{\psi(\gamma)}{np_1^{\nu_1} \cdots p_r^{\nu_r}} \right\},$$

and the measure of their overlap will be at most

$$2 \min \left\{ \frac{\psi(\beta)}{mp_1^{\mu_1} \cdots p_r^{\mu_r}}, \frac{\psi(\gamma)}{np_1^{\nu_1} \cdots p_r^{\nu_r}} \right\}$$

(since the worst-case scenario is that one interval is completely contained inside the other).

Similarly, the p_i -adic intervals will overlap when

$$\left| \frac{a}{m} - \frac{b}{n} \right|_{p_i} \leq \max\{p_i^{\mu_i} \psi(\beta), p_i^{\nu_i} \psi(\gamma)\},$$

and the measure of their overlap will be at most

$$\min\{p_i^{\mu_i} \psi(\beta), p_i^{\nu_i} \psi(\gamma)\}.$$

So if we write

$$\Delta_\infty = 2 \max \left\{ \frac{\psi(\beta)}{mp_1^{\mu_1} \cdots p_r^{\mu_r}}, \frac{\psi(\gamma)}{np_1^{\nu_1} \cdots p_r^{\nu_r}} \right\}, \quad \delta_\infty = 2 \min \left\{ \frac{\psi(\beta)}{mp_1^{\mu_1} \cdots p_r^{\mu_r}}, \frac{\psi(\gamma)}{np_1^{\nu_1} \cdots p_r^{\nu_r}} \right\}$$

and

$$\Delta_{p_i} = \max\{p_i^{\mu_i} \psi(\beta), p_i^{\nu_i} \psi(\gamma)\}, \quad \delta_{p_i} = \min\{p_i^{\mu_i} \psi(\beta), p_i^{\nu_i} \psi(\gamma)\},$$

then we have

$$\lambda(A_\beta \cap A_\gamma) \leq \left(\delta_\infty \prod_{i=1}^r \delta_{p_i} \right) N(\beta, \gamma),$$

where

$$N(\beta, \gamma) := \left\{ (a, b) \mid \begin{array}{l} 1 \leq a \leq m, 1 \leq b \leq n, \\ |an - bm|_\infty \leq mn\Delta_\infty, |an - bm|_{p_i} \leq \Delta_{p_i} \text{ for } i = 1, \dots, r \end{array} \right\}.$$

Now, since $|\cdot|_{p_i}$ only takes values which are powers of p_i , for each p_i we find the $\tau_i \in \mathbb{Z}$ such that

$$p_i^{-\tau_i} \leq \Delta_{p_i} < p_i^{1-\tau_i},$$

and then consider the individual cases

$$|an - bm|_{p_i} = p_i^{-t_i}$$

for $t_i \geq \tau_i$.

If

$$|an - bm|_{p_1} = p_1^{t_1}, \dots, |an - bm|_{p_r} = p_r^{t_r},$$

then we have

$$an - bm = p_1^{t_1} \cdots p_r^{t_r} k$$

for some $k \in \mathbb{Z} - p_1\mathbb{Z} - \cdots - p_r\mathbb{Z}$. So we get

$$N(\beta, \gamma) = \sum_{t_1 \geq \tau_1} \cdots \sum_{t_r \geq \tau_r} \sum_{\substack{k=1 \\ p_i \nmid k \text{ for} \\ i=1, \dots, r}}^{\frac{mn\Delta_\infty}{p_1^{t_1} \cdots p_r^{t_r}}} \# \left\{ (a, b) \mid \begin{array}{l} 1 \leq a \leq m, \ 1 \leq b \leq n, \\ |an - bm|_\infty = p_1^{t_1} \cdots p_r^{t_r} k \end{array} \right\}.$$

We now use the following standard lemma, from elementary number theory:

Lemma 5.2. *Suppose that $m, n \in \mathbb{N}$ and $x \in \mathbb{Z}$. Then the equation*

$$an - bm = x$$

has solutions $a, b \in \mathbb{Z}$ if and only if $(m, n) \mid x$. If (a_0, b_0) is a particular solution, then the set of all solutions is

$$\left\{ \left(a_0 + \frac{\ell m}{(m, n)}, b_0 + \frac{\ell n}{(m, n)} \right) \mid \ell \in \mathbb{Z} \right\}.$$

We see that there are at most (m, n) solutions with $1 \leq a \leq m$ and $1 \leq b \leq n$. Hence we have

$$\begin{aligned} \sum_{\substack{k=1 \\ p_i \nmid k \text{ for} \\ i=1, \dots, r}}^{\frac{mn\Delta_\infty}{p_1^{t_1} \cdots p_r^{t_r}}} \# \left\{ (a, b) \mid \begin{array}{l} 1 \leq a \leq m, \ 1 \leq b \leq n, \\ |an - bm|_\infty = p_1^{t_1} \cdots p_r^{t_r} k \end{array} \right\} &\leq 2 \sum_{\substack{k=1 \\ p_i \nmid k \text{ for} \\ i=1, \dots, r \\ (m, n) \mid p_1^{t_1} \cdots p_r^{t_r} k}}^{\frac{mn\Delta_\infty}{p_1^{t_1} \cdots p_r^{t_r}}} (m, n) \\ &\leq 2(m, n) \sum_{\substack{k=1 \\ (m, n) \mid p_1^{t_1} \cdots p_r^{t_r} k}}^{\frac{mn\Delta_\infty}{p_1^{t_1} \cdots p_r^{t_r}}} 1 \\ &\leq 2(m, n) \frac{\frac{mn\Delta_\infty}{p_1^{t_1} \cdots p_r^{t_r}}}{(m, n)} \\ &= \frac{2mn\Delta_\infty}{p_1^{t_1} \cdots p_r^{t_r}}, \end{aligned}$$

and therefore

$$\begin{aligned} N(\beta, \gamma) &\leq 2mn\Delta_\infty \sum_{t_1 \geq \tau_1} \cdots \sum_{t_r \geq \tau_r} \frac{1}{p_1^{t_1} \cdots p_r^{t_r}} \\ &\leq 2mn\Delta_\infty \prod_{i=1}^r \left(\sum_{t_i \geq \tau_i} \frac{1}{p_i^{t_i}} \right) \\ &\leq 4mn\Delta_\infty \prod_{i=1}^r p_i^{-\tau_i} \\ &\leq 4mn\Delta_\infty \prod_{i=1}^r \Delta_{p_i}. \end{aligned}$$

So we get

$$\begin{aligned}
\lambda(A_\beta \cap A_\gamma) &\leq \left(\delta_\infty \prod_{i=1}^r \delta_{p_i} \right) \left(4mn \Delta_\infty \prod_{i=1}^r \Delta_{p_i} \right) \\
&= 4mn \Delta_\infty \delta_\infty \prod_{i=1}^r \Delta_{p_i} \delta_{p_i} \\
&= 4mn \cdot 4 \frac{\psi(\beta)\psi(\gamma)}{mn p_1^{\mu_1} p_1^{\nu_1} \cdots p_r^{\mu_r} p_r^{\nu_r}} \cdot \prod_{i=1}^r p_i^{\mu_i} p_i^{\nu_i} \psi(\beta)\psi(\gamma) \\
&= 16\psi(\beta)^{r+1} \psi(\gamma)^{r+1}
\end{aligned}$$

as required. \square

6. PROOF OF THEOREM 1.2

In §3, we defined sets $\mathcal{A}_\gamma(\psi)$ such that

$$\mathcal{A}(\psi) = \limsup_{\ell(\gamma) \rightarrow \infty} \mathcal{A}_\gamma(\psi),$$

and showed that these sets could be written as

$$\mathcal{A}_\gamma(\psi) = \bigcup_{\substack{a=1 \\ (a,n)=1}}^n \left(B_\infty \left(\frac{a}{n}, \frac{\psi(\gamma)}{n p_1^{\nu_1} \cdots p_r^{\nu_r}} \right) \times \prod_{i=1}^r B_{p_i} \left(\frac{a}{n}, p_i^{\nu_i} \psi(\gamma) \right) \right).$$

Now we need to show that

$$\limsup_{\ell(\gamma) \rightarrow \infty} \mathcal{A}_\gamma(\psi)$$

has measure 1. Since Lemma 1.3 states that $\mathcal{A}(\psi)$ has either measure 0 or measure 1, we only need to show that this set has *positive* measure.

We use a lemma (Lemma 2.3 from [4], which we quote below) to get a lower bound on the size of our limsup set.

Lemma 6.1. *Let X be a measure space with measure λ such that $\lambda(X)$ is finite. Let \mathcal{E}_n be a sequence of measurable subsets of X such that*

$$\sum_{n=1}^{\infty} \lambda(\mathcal{E}_n) = \infty.$$

Then the set E of points belonging to infinitely many sets \mathcal{E}_n satisfies

$$\lambda(E) \geq \limsup_{N \rightarrow \infty} \left(\sum_{n=1}^N \lambda(\mathcal{E}_n) \right)^2 \left(\sum_{m,n=1}^N \lambda(\mathcal{E}_m \cap \mathcal{E}_n) \right)^{-1}.$$

We note that $[0, 1) \times \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_r}$ is such a measure space, and we consider our measurable subsets \mathcal{A}_γ . We have that

$$\frac{2\varphi(n)\psi(\gamma)^{r+1}}{n p_1 \cdots p_r} < \lambda(\mathcal{A}_\gamma) \leq \frac{2\varphi(n)\psi(\gamma)^{r+1}}{n},$$

and hence if

$$\sum_{\gamma \in \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right]} \frac{\varphi(n) \psi(\gamma)^{r+1}}{n} = \infty,$$

then we have that

$$\sum_{\gamma \in \mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right]} \lambda(\mathcal{A}_\gamma) = \infty$$

as well. So as in §3, we consider an ordering γ_i of $\mathbb{Z} \left[\frac{1}{p_1}, \dots, \frac{1}{p_r} \right]$ such that if $i \leq j$, then $\ell(\gamma_i) \leq \ell(\gamma_j)$. Then we have, by the lemma, that

$$\lambda(\limsup \mathcal{A}_\gamma) \geq \limsup_{N \rightarrow \infty} \left(\sum_{i=1}^N \lambda(\mathcal{A}_{\gamma_i}) \right)^2 \left(\sum_{i,j=1}^N \lambda(\mathcal{A}_{\gamma_i} \cap \mathcal{A}_{\gamma_j}) \right)^{-1}.$$

Now consider a subsequence $\{i_N\}$ of \mathbb{N} defined by

$$i_N := \max\{i \in \mathbb{N} \mid \ell(\gamma_i) = N\}.$$

Then we have

$$\begin{aligned} \lambda(\limsup \mathcal{A}_\gamma) &\geq \limsup_{N \rightarrow \infty} \left(\sum_{i=1}^N \lambda(\mathcal{A}_{\gamma_i}) \right)^2 \left(\sum_{i,j=1}^N \lambda(\mathcal{A}_{\gamma_i} \cap \mathcal{A}_{\gamma_j}) \right)^{-1} \\ &\geq \limsup_{N \rightarrow \infty} \left(\sum_{i=1}^{i_N} \lambda(\mathcal{A}_{\gamma_i}) \right)^2 \left(\sum_{i,j=1}^{i_N} \lambda(\mathcal{A}_{\gamma_i} \cap \mathcal{A}_{\gamma_j}) \right)^{-1} \\ &= \limsup_{N \rightarrow \infty} \left(\sum_{\ell(\gamma) \leq N} \lambda(\mathcal{A}_\gamma) \right)^2 \left(\sum_{\ell(\beta), \ell(\gamma) \leq N} \lambda(\mathcal{A}_\beta \cap \mathcal{A}_\gamma) \right)^{-1}. \end{aligned}$$

So now we need to show that

$$\limsup_{N \rightarrow \infty} \left(\sum_{\ell(\gamma) \leq N} \lambda(\mathcal{A}_\gamma) \right)^2 \left(\sum_{\ell(\beta), \ell(\gamma) \leq N} \lambda(\mathcal{A}_\beta \cap \mathcal{A}_\gamma) \right)^{-1}$$

is positive. We can do this by appealing to the overlap estimates from Lemma 5.1. Using this, we get

$$\begin{aligned}
\frac{\left(\sum_{\ell(\gamma) \leq N} \lambda(\mathcal{A}_\gamma)\right)^2}{\sum_{\ell(\beta), \ell(\gamma) \leq N} \lambda(\mathcal{A}_\beta \cap \mathcal{A}_\gamma)} &\geq \frac{\left(\sum_{\ell(\gamma) \leq N} \frac{2\varphi(n)\psi(\gamma)^{r+1}}{np_1 \cdots p_r}\right)^2}{C \sum_{\ell(\beta), \ell(\gamma) \leq N} \psi(\gamma)^{r+1} \psi(\beta)^{r+1}} \\
&\geq C \frac{\left(\sum_{\ell(\gamma) \leq N} \frac{\varphi(n)\psi(\gamma)^{r+1}}{n}\right)^2}{\left(\sum_{\ell(\gamma) \leq N} \psi(\gamma)^{r+1}\right)^2} \\
&= C \left(\frac{\sum_{\ell(\gamma) \leq N} \frac{\varphi(n)\psi(\gamma)^{r+1}}{n}}{\sum_{\ell(\gamma) \leq N} \psi(\gamma)^{r+1}} \right)^2,
\end{aligned}$$

and hence since we assumed

$$\limsup_{N \rightarrow \infty} \frac{\sum_{\ell(\gamma) \leq N} \frac{\varphi(n)\psi(\gamma)^{r+1}}{n}}{\sum_{\ell(\gamma) \leq N} \psi(\gamma)^{r+1}} > 0,$$

we have

$$\limsup_{N \rightarrow \infty} \left(\sum_{\ell(\gamma) \leq N} \lambda(\mathcal{A}_\gamma) \right)^2 \left(\sum_{\ell(\beta), \ell(\gamma) \leq N} \lambda(\mathcal{A}_\beta \cap \mathcal{A}_\gamma) \right)^{-1} > 0,$$

and hence $\lambda(\mathcal{A}(\psi)) = 1$ as required.

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